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# Maximal Partial Spreads and Translation Nets of Small Deficiency

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Partial $t$-spreads and translation nets of small deficiency are considered (meaning $s > (d-1)^2$, where $s$ is the order and $d$ the deficiency). It is proved that (almost) all such translation nets do in fact come from partial $t$-spreads and some open questions about the nets belonging to maximal partial $t$-spreads are answered. Some new examples of maximal partial $t$-spreads and of unimbeddable nets of small deficiency are also constructed.

## 1. INTRODUCTION

One of the central problems of combinatorial theory is the precise determination of the maximum number $N(s)$ of mutually orthogonal Latin squares of order $s$ (which includes the determination of the orders of finite projective planes). The only known values are $N(6) = 1$ and $N(s) = s - 1$ for prime powers $s$; and the only non-trivial upper bound on $N(s)$ is given by Bruck's famous completion theorem in conjunction with the Bruck–Ryser theorem on the non-existence of certain projective planes. (For background information and undefined terms, the reader may consult Dembowski [12].) Let us recall Bruck's results (in terms of nets instead of Latin squares):

1.1. THEOREM (Bruck [6]). *Let* **D** *be a net of order* $s$*, degree* $r$ *and deficiency* $d = s + 1 - r$*. Assume that* **D** *can be imbedded into an affine plane* **A** *of order* $s$ *and*

$$s > (d-1)^2. \tag{1.1a}$$

*Then* **A** *is uniquely determined. Moreover, a sufficient condition for* **D** *to be imbeddable is*

$$p(d-1) < s \tag{1.1b}$$

*where the polynomial* $p$ *is defined by*

$$p(x) = \tfrac{1}{2}x^4 + x^3 + x^2 + \tfrac{3}{2}x. \tag{1.1c}$$



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This poses the question of how good the bound (1.1b) is. The first examples of unimbeddable nets of small deficiency have been obtained by Bruen [8] using algebraic methods (i.e., partial spreads and translation nets). It is the aim of this paper to provide some further examples and to answer some of the questions on partial spreads left open in Bruen's paper. Before giving more precise information on the results achieved we will introduce some terminology and review the present knowlege on this problem.

1.2. DEFINITION. Let **D** be a net of order $s$ and deficiency $d$. **D** is called *complete if* $d=0$. We say that **D** is a *net of small deficiency* if (1.1a) is satisfied. **D** is called *extendible* if it can be imbedded into a net $\mathbf{D}'$ of deficiency $<d$, and *maximal* otherwise. If **D** may be imbedded into a complete net, it is called *imbeddable.*

1.3. DEFINITION. A net **D** is called a *translation net* if it admits a point-regular collineation group $G$ fixing each parallel class; $G$ is called the *translation group* of **D**.

1.4. DEFINITION. Let $G$ be a group of order $s^2$ and let $\mathbf{U}=\{U_1,..., U_r\}$ be a set of $r$ subgroups of order $s$ of $G$ satisfying

$$U_i \cap U_j = \{0\} \qquad \text{whenever } i \neq j \tag{1.4a}$$

(where we write $G$ additively). Then **U** is called a *partial congruence partition* of *degree r*, *order s* and *deficiency* $d=s+1-r$ in $G$ (or, more briefly, an $(s,r)-$pcp). If in fact $G$ is the additive group of the $(2t+2)$-dimensional vector space $V(2t+2,q)$ over the field $GF(q)$ and if all components $U_i$ of **U** are $(t+1)$-dimensional subspaces of $V(2t+2,q)$, then **U** is called a *partial t-spread* over $GF(q)$. In this case, we may also view **U** as a collection of mutually skew $t$-dimensional subspaces of the projective space $PG(2t+1,q)$.

Translation nets and pcp's have been studied by Sprague [26] and the present author [16]; they turn out to be the same, basically:

1.5. LEMMA. *Let* **U** *be an* $(s,r)$-pcp *in* $G$ *and define an incidence structure* $\mathbf{D}(\mathbf{U})$ *as follows*:

$$\mathbf{D}(\mathbf{U}) = (G, \{U+x\colon U\in \mathbf{U}, x\in G\}, \in). \tag{1.5a}$$

*Then* $\mathbf{D}(\mathbf{U})$ *is a translation net of order s and degree r. Moreover, every translation net may be represented in this way.*

We remark that this goes back to André [1], who considered only the case of complete translation nets, i.e., of affine translation planes. The term "translation net" in general has often been used only for nets corresponding to a partial $t$-spread, cf., e.g., Bruen [9]. As already mentioned, Bruen's examples of unimbeddable nets of small deficiency are constructed from partial 1-spreads. One might then try to use other pcp's to obtain more examples. However, this turns out to be impossible: In Section 2, we will show that any translation net of small deficiency has elementary abelian translation group (and thus the corresponding pcp may be viewed as a partial $t$-spread over $GF(p)$ for some suitable $t$), excepting two trivial cases for $s = 2$ or $s = 4$.

We now list the known results on *maximal* partial $t$-spreads (the terminology of 1.2 for nets will be used analogously for pcp's and partial $t$-spreads too); unless stated otherwise, all maximal nets, partial $t$-spreads, etc., will be assumed to be incomplete henceforth.

1.6. *Result.* Let $\mathbf{U}$ be a maximal partial $t$-spread of degree $r$ over $GF(q)$. For $t = 1$ one has

$$2q \leqslant r \leqslant q^2 - \sqrt{q} \tag{1.6a}$$

(Glynn [15], Mesner [20]); if $q$ is not a square, the upper bound may be improved to

$$p(d-1) \geqslant q^2 \tag{1.6b}$$

(Bruen [9]). For $t > 1$, one has

$$q + \sqrt{q} \leqslant r \leqslant q^{t+1} - \sqrt{q}, \tag{1.6c}$$

and for $q \geqslant 4$ the lower bound in (1.6c) may be improved to $q + \sqrt{q} + 1$ (Beutelspacher [4], Bruen [10]). (Beutelspacher [2] has asserted a stronger upper bound than (1.6c), but this turned out to be erroneous, see [3].) Many examples of maximal partial $t$-spreads are known; we will only list those having small deficiency:

1.7. *Result.* A maximal partial $t$-spread of deficiency $d$ over $GF(q)$ is known to exist in all of the following cases:

$t = 1$, $d = q$ for all prime powers $q \geqslant 3$ (Bruen [7]); (1.7a)

$t = 1$, $d = q - 1$ for all prime powers $q \geqslant 4$ (Bruen [7], Bruen and Thas [11], Freeman [14]); (1.7b)

$t = 2a + 1$ with $a \geqslant 1$, $d = q^{a+1}$ for all prime powers $q \geqslant 4$ (Beutelspacher [4]). (1.7c)

In Section 6, we will give a new proof for (1.7b) for those $q$ which are not primes; this may be interesting as the construction of Freeman (which covers the case of even powers of 2) is rather involved. In Section 7 we will obtain examples of maximal partial $t$-spreads for $t = 2a + 1$ having deficiency $q^{a+1} - 1$. Finally, we mention the known examples of unimbeddable nets of small deficiency:

1.8. *Result* (Bruen [8]). An unimbeddable net of small deficiency and order $p^2$ exists for all odd primes $p$. In fact one may always take $d = p$ and for $p \geqslant 5$ also $d = p - 1$. The cases $p = d = 3$ and $p = 5$, $d = 4$ show that (1.1b) is best possible for $s = 9$ and $s = 25$.

It is an open question whether the nets of 1.8 are maximal or not. Answering some questions of Bruen [8] we will prove the following results. Any maximal partial 1-spread of small deficiency over $GF(p)$ yields a maximal net; hence we obtain maximal nets of deficiency $p$ and $p - 1$ from (1.7a, b). Any maximal partial 1-spread of deficiency $q$ over $GF(q)$ yields an unimbeddable net (though we have not been able to determine the exact deficiency in this case) and thus unimbeddable nets of small deficiency exist also for all orders $s = q^2$, $q$ a prime power (using (1.7a)). Regarding the case $s = q^2$, $d = q - 1$ we construct maximal partial 1-spreads in this situation which nevertheless yield imbeddable nets, in fact translation planes. We also show that any imbeddable translation net of small deficiency is imbeddable in a translation plane which provides a simple proof for a result of Bruen [9]. As already mentioned we also construct new examples of maximal partial $t$-spreads.

## 2. A Non-Existence Result for Translation Nets

In this section we show that translation nets of small deficiency correspond to partial $t$-spreads (with two exceptions).

2.1. Theorem. *Let* **D** *be a translation net with small deficiency. Then the translation group* $G$ *of* **D** *is elementary abelian, unless* **D** *has order* $s = 2$ *or* 4.

*Proof.* The proof rests mainly on a result from [16] (note that our $(s, r)$-pcp's have been called $(s, r; 1)$-pcp's there): $G$ is elementary abelian provided that

$$r > \left[\frac{2s - 2}{p + 1}\right] + 2 \qquad (p \text{ the smallest prime divisor of } s) \qquad (2.1\text{a})$$

where $[x]$ denotes the largest integer not exceeding $x$ (Corollary 5.6 of [16], where the bound has been misprinted with $2s - 1$ instead of $2s - 2$). Now (1.1a) is satisfied, as **D** has small deficiency, and thus we have in fact

$$r > s - \sqrt{s}. \qquad (2.1\text{b})$$

Therefore (2.1a) is certainly satisfied provided that

$$s - \sqrt{s} \geqslant \frac{2s-2}{3} + 2 \Leftrightarrow s \geqslant 3\sqrt{s} + 4$$

and our assertion holds for all $s \geqslant 16$ and also (trivially) for the primes $s = 3, 5, 7, 11, 13$. Each of the composite values $s = 6, 10, 12, 14, 15$ yields only groups $G$ with a normal Sylow subgroup $S$; but then one would also have pcp's with the same value of $r$ in both $S$ and $G/S$ (see Sprague [26, Proposition 2.5]) which contradicts (2.1b). If $s$ is a prime power, say, $s = p^a$, and $G$ is abelian but not elementary abelian, then $r \leqslant p^{a/2} + 1$ by Corollary 5.4 of [16]; hence for $s = 8$, either $G$ is elementary abelian or non-abelian, and the latter case is excluded by the results in Section 4 of Sprague [26]. Finally, the case $s = 9$ may be excluded by another application of (2.1a) with $p = 3$ here (or by Corollary 4.11 of [16]). ▮

2.2. Corollary. *Let* **D** *be a translation net of small deficiency. Then the order of* **D** *is a prime power (say, $p^a$) and* $\mathbf{D} = \mathbf{D}(\mathbf{U})$ *for a partial t-spread* **U** *over $GF(p)$, unless $p^a = 2$ or 4.*

2.3. Example. The exceptional cases in 2.2 and 2.1 are truly exceptional: For $s = 2$, take $G = \mathbb{Z}_4$ (the group of residues mod 4) and $\mathbf{U} = \{0, 2\}$. For $s = 4$, take $G = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ and $U_1 = \{0\} \times \mathbb{Z}_4$, $U_2 = \mathbb{Z}_4 \times \{0\}$, $U_3 = \{(x, x) : x \in \mathbb{Z}_4\}$ (this is, by the way, a maximal net).

2.4. *Remark.* It is clear that results similar to 2.1 may be proved for all conditions of the form

$$r > s - s^{\delta} \tag{2.4a}$$

where $\delta < 1$, using (2.1a) again. The determination of the exceptional values of $s$ will become more difficult, of course, if $\delta$ becomes larger than $\frac{1}{2}$. But no construction of infinite series of translation nets with non-elementary abelian groups and deficiency in the order of magnitude of $s^{\delta}$ is possible for $\delta < 1$.

## 3. Imbeddings of Translation Nets with Small Deficiency

All results of this section rest on the following simple but fundamental observation:

3.1. Lemma. *Let* **D** *be a net of small deficiency and assume that* **D** *is imbeddable in an affine plane* **A**. *Then* Aut **D** $\leqslant$ Aut **A**.

*Proof.* By a well-known result of Bruck [6], **A** is obtained from **D** by adjoining all transversals of **D** to **D**. But if $T$ is a transversal of **D**, then so is $T^{\gamma}$ for all $\gamma \in \operatorname{Aut} \mathbf{D}$; hence also $\gamma \in \operatorname{Aut} \mathbf{A}$. ▮

3.2. THEOREM. *Let* **D** *be a translation net of small deficiency and assume that* **D** *is imbedded into an affine plane* **A** (*which is unique by Theorem* 1.1). *Then* **A** *is a translation plane.*

*Proof.* Let $G$ be the translation group of **D**; then $G \leqslant \operatorname{Aut} \mathbf{A}$ by Lemma 3.1. We will show that $G$ is actually also the translation group of **A**. Let **P** be the projective extension of **A**. It is well known that any collineation of **P** fixes equally many points and lines. Now consider any $\gamma \in G$ which is contained in one of the components of the pcp **U** corresponding to **D**. As a collineation of **P** $\gamma$ has precisely $s+1$ fixed lines (where $s$ is the order of **D**). If $\gamma$ lies in the component $U$ of **U**, then $\gamma$ fixes the $s$ lines parallel to $U$, as $G$ is abelian by Theorem 2.1, and also the line at infinity. Thus $\gamma$ also has $s+1$ fixed points in **P**; none of these points can be an affine point, and so $\gamma$ fixes all points on the line at infinity, i.e. $\gamma$ is a translation of **A**. Hence the components of **U** consist of translations of **A**, and as $G$ is the product of any two of these components, all elements of $G$ are translations of **A**. ▮

3.3. COROLLARY. *Let* **D** *be a translation net of order s and deficiency d, and assume that* (1.1b) *is satisfied. Then* **D** *may be imbedded into an affine translation plane.*

We remark that Bruen has proved 3.3 under the extra assumption that **D** corresponds to a partial $t$–spread in [9] (of course this assumption is no real restriction in view of Corollary 2.2). However, the proof presented here is much simpler (at least if one assumes the fact that $G$ is abelian). Another immediate consequence of Theorem 3.2 is the following result (which has been proved by Bruen [8] for two special classes of partial 1-spreads):

3.4. THEOREM. *Let* **U** *be a maximal partial t-spread of small deficiency over* $GF(p)$. *Then the translation net* **D**(**U**) *is unimbeddable.*

Using 3.4 in conjunction with (1.7c) we thus have a new class of unimbeddable nets of small deficiency:

3.5. COROLLARY. *There exists a unimbeddable translation net of order* $p^{2a}$ ($a \geqslant 1$) *and deficiency* $p^a$ *for all primes* $p \geqslant 5$.

We will give another proof for this result (which will also eliminate the restriction on $p$) in Section 5. Applying 3.4 to the maximal partial spreads of Bruen [7] (see (1.7a, b)) we obtain a simple alternative proof for his result

1.8. In the next section we will show that the examples in 1.8 are in fact already maximal (though we cannot prove this for the examples just constructed in 3.5).

## 4. Some Maximal Nets of Small Deficiency

We begin with a general Lemma regarding the extendibility of translation nets:

4.1. Lemma (Ostrom). *Let* $\mathbf{U}$ *be a pcp in a group* $G$ *and assume that at least one component of* $\mathbf{U}$ *is a normal subgroup of* $G$. *Then* $\mathbf{D}(\mathbf{U})$ *is extendible (as a net) iff it has a transversal.*

*Proof.* Let $U$ be a normal component of $\mathbf{U}$; then the elements of $U$ all fix the lines of $\mathbf{D}(\mathbf{U})$ parallel to $U$. Thus $U$ consists (in the terminology of Ostrom [23]) of "strict translations" of $\mathbf{D}(\mathbf{U})$ and thus our Lemma is a special case of Theorem 4 of Ostrom [23]. (In fact it is not difficult to see that the set $\mathbf{T} = \{T + u : u \in U\}$ may be added as a further parallel class to $\mathbf{D}(\mathbf{U})$ whenever $T$ is a transversal.) ∎

4.2. Lemma. *Let* $\mathbf{U}$ *be a maximal* $(s, r)$*-pcp in the group* $G$, *where at least one component of* $\mathbf{U}$ *is a normal subgroup of* $G$. *Assume that* $\mathbf{D} = \mathbf{D}(\mathbf{U})$ *is not maximal. Then* $\mathbf{D}$ *may be extended by* $x$ *parallel classes where* $x$ *is some divisor* $\neq 1$ *of* $s$. *Moreover, the resulting net* $\mathbf{D}'$ *still admits* $G$ *as a collineation group (though not as translation group).*

*Proof.* Let $T$ be any transversal of $\mathbf{D}$ and define $\mathbf{T}$ as in the proof of 4.1; hence $\mathbf{T}$ consists of $s$ pairwise disjoint transversals of $\mathbf{D}$. Hence the $T$-orbit of $G$ has length $sx$ for some divisor $x$ of $s$. Here $x \neq 1$ as otherwise $\mathbf{U} \cup \{T\}$ would be a pcp strictly larger than $\mathbf{U}$ (with $0 \in T$). The $sx$ transversals obtained thus split naturally into $x$ classes consisting of $s$ pairwise disjoint transversals of $\mathbf{D}$ each; thus each of these classes may be added as a further parallel class. We still have to show that transversals from distinct such classes intersect in precisely one point. But this is an easy consequence of a result of Bruck [6] which asserts that any two transversals of a net of small dificiency have at most one point in common. ∎

4.3. Theorem. *Let* $\mathbf{U}$ *be a maximal partial* 1*-spread of small deficiency* $d$ *over* $GF(p)$. *Then* $\mathbf{D}(\mathbf{U})$ *is a maximal (translation) net of deficiency* $d$.

*Proof.* Note that $s = p^2$ and thus $d \leqslant p$ under our assumptions. Assume that $\mathbf{D}(\mathbf{U})$ is not maximal; then we may add $p$ parallel classes to $\mathbf{D}(\mathbf{U})$ by

Lemma 4.2. Thus $\mathbf{D}(\mathbf{U})$ can be imbedded into an affine plane which is a translation plane by Theorem 3.2. This contradicts the maximality of $\mathbf{U}$. ∎

Applying Theorem 4.3 to (1.7a, b) we have at once:

4.4. COROLLARY (Bruen). *Let $p$ be an odd prime. Then there exists a maximal (translation) net of order $p^2$ and deficiency $p$. If $p \geqslant 5$, then there also exists a maximal (translation) net of deficiency $p-1$ and order $p^2$.*

Note that Theorem 4.3 answers the question posed in Comment 2 of Bruen [8]: His examples of unimbeddable nets are in fact all maximal. Bruen has also obtained this result in [9] using a different method of proof.

4.5. *Remark.* After I had completed this research, David A. Drake kindly drew my attention to a result related to Corollary 4.4 which has recently been obtained by one of his students and is as yet unpublished. In his Ph.D. thesis [13] Dow constructs maximal nets of order $q^2$ and deficiency $q$ whenever $q$ is a prime power. He obtains his examples by adjoining to the net $\mathbf{D}_0$ consisting of the $q^2 - q$ parallel classes of lines in common between the desarguesian affine plane $AG(2, q)$ and its derived plane, the Hall plane $H(q)$, a further parallel class containing lines of both these planes. The resulting net is, however, no longer a translation net (though $\mathbf{D}_0$ is a translation net, of course). In the next section, we will obtain maximal translation nets of order $q^2$ and deficiency at most $q$ for all prime powers $q$ (but we cannot determine their exact deficiency).

## 5. SOME UNIMBEDDABLE MAXIMAL PARTIAL SPREADS

In his Comment 1 Bruen [8] posed the question whether or not the maximal partial 1-spreads of deficiency $q$, resp. $q-1$, which he had constructed in [7] gave rise to unimbeddable translation nets (assuming $q$ to be a prime power which is not a prime). We will now give a partial answer to his question by proving that $\mathbf{D}(\mathbf{U})$ is unimbeddable for every maximal partial 1-spread of deficiency $q$. In the next section we will construct many examples of maximal partial 1-spreads $\mathbf{U}$ of deficiency $q-1$ for which $\mathbf{D}(\mathbf{U})$ is imbeddable (though we do not know whether these include the examples of Bruen).

5.1. THEOREM. *Let $\mathbf{U}$ be a maximal partial 1-spread of deficiency $q$ over $GF(q)$. Then the translation net $\mathbf{D} = \mathbf{D}(\mathbf{U})$ is unimbbedable.*

*Proof.* Let $U_1, \dots, U_{q^2-q+1}$ be the components of $\mathbf{U}$. Thus the $U_i$ are 2-dimensional subspaces of the vector space $V(4, q)$ and the additive group $G$ of this vector space is the translation group of $\mathbf{D}$. Assume that $\mathbf{D}$ may be imbedded into an affine plane **A**. Then **A** is a translation plane by Theorem 3.2 and therefore $\mathbf{A} = \mathbf{D}(\mathbf{W})$ for some pcp **W** in $G$. We may obtain **W** by adjoining to **U** $q$ further subgroups $U_{q^2-q+2}, \ldots, U_{q^2+1}$ of order $q^2$ of $G$. By the maximality of **U**, the new components will not be subspaces of $V(4, q)$ which amounts to saying that they are not fixed under the dilatation group $H$ of **D** (which of course is isomorphic to $GF(q)^*$: if it were isomorphic to $GF(q^2)^*$ then **U** would not be maximal). We plan to obtain a contradiction by proving that $H$ fixes at least one of the new components.

Now let $\lambda$ be any element of $GF(q)^*$ and denote by $\bar{\lambda}$ the corresponding dilatation of **D** (i.e., $\bar{\lambda}$: $x \to x\lambda$ for $x \in V(4, q)$). By Lemma 3.1 $\bar{\lambda}$ is a collineation of **A**; hence $H$ acts on the set of new components of **W** and fixes none of them. For any new component $U$ of **W** define

$$F(U) := \{\lambda \in GF(q)^* : \bar{\lambda} \text{ fixes } U\} \cup \{0\} \tag{5.1a}$$

and note that $F(U)$ is a subfield of $GF(q)$ containing $GF(p)$ (where $q$ is a power of the prime $p$). Note further that the number of components of **W** fixed by $\bar{\lambda}$ is equal to the number of fixed points of $\bar{\lambda}$ on the line at infinity of **A** and is at least $q^2 - q + 1$. We now restrict our attention to elements $\lambda$ of order $r^a$ for some prime $r$. Then the number $f(\lambda)$ of fixed points of $\bar{\lambda}$ on the line at infinity is congruent to $q^2 + 1$ modulo $r$ and thus (as $r^a$ divides $q - 1$) congruent to 2 modulo $r$. But $q^2 - q + 1$ is congruent to 1 modulo $r$ and therefore $\bar{\lambda}$ fixes at least one new component of **W**. But if $r^a$ is any prime power divisor of $q - 1$ there exists an element $\lambda$ of order $r^a$ as $GF(q)^*$ is cyclic; hence there also exists some new component $U$ for which $F(U)$ contains both $GF(p)$ and an element of order $r^a$. We will use this fact to show that $F(U) = GF(q)$ for at least one new component $U$ which will be the desired contradiction. We shall consider some special cases first.

Let $q = p^n$ and assume first that $n = 2$. Then we may choose a $\lambda \notin GF(p)^*$ of prime order $r$ dividing $p + 1$; hence $F(U)$ is a field strictly larger than $GF(p)$ and therefore equal to $GF(q)$. Next assume that $p = 2$ and $n = 6$: Then we may choose $r^a = 9$ and obtain for $F(U)$ a subfield of $GF(2^6)$ containing an element of order 9, i.e., $GF(2^6)$ itself. Finally, consider any of the remaining cases. Then Zsigmondy's theorem ([27]; see also Lüneburg [19] for a simple proof of this result) ensures the existence of a prime $r$ dividing $q - 1 = p^n - 1$ but not dividing $p^x - 1$ for any $x < n$. For this $r$ again $F(U) = GF(q)$, completing our proof. ∎

5.2. Corollary. *For any prime power $q$ there exists an unimbeddable translation net of deficiency $q$ and order $q^2$.*

*Proof.* For $q \geqslant 3$ apply Theorem 5.1 to the partial 1-spreads of (1.7a). For $q = 2$ an example was given in 2.3. ∎

5.3. *Remark.* The nets of 5.2 may then be extended to maximal nets of deficiency at most $q$; we do not know, however, whether or not they are already maximal. Regarding examples of maximal nets of order $q^2$ and deficiency $q$, cf. Remark 4.5.

We conclude this section by observing that the proof of Theorem 5.1 carries over to partial $t$-spreads over $GF(q)$ for values $t$ of the form $2a+1$. One then obtains:

5.4. THEOREM. *Let* $\mathbf{U}$ *be a maximal partial t-spread* (*with* $t = 2a+1$) *of deficiency* $q^{a+1}$ *over* $GF(q)$. *Then the translation net* $\mathbf{D}(\mathbf{U})$ *is unimbeddable.*

Theorem 5.4 shows that the partial $t$-spreads of Beutelspacher mentioned in (1.7c) also yield unimbeddable translation nets (but of course no examples with parameters not yet obtained in Corollary 5.2).

## 6. SOME IMBEDDABLE MAXIMAL PARTIAL SPREADS

In this section we will construct maximal partial 1-spreads of deficiency $q-1$ over $GF(q)$ (where $q$ is not a prime) which yield imbeddable translation nets. To motivate our construction we will first consider the method of Bruen [7] for obtaining maximal partial 1-spreads over $GF(q)$. Bruen starts with a complete 1-spread $\mathbf{U}$ over $GF(q)$ from which he first removes $q+1$ components and then adds one or two further components in such a way that the resulting partial 1-spread $\mathbf{U}_0$ is maximal (and then of deficiency $q$ of $q-1$). Thus his examples are in fact constructed from translation planes $\mathbf{A}$ ($= \mathbf{D}(\mathbf{U})$) of order $q^2$ and dimension 2 over their kernel. Now assume that $\mathbf{D}(\mathbf{U}_0)$ can be imbedded into an affine plane $\mathbf{A}'$; then $\mathbf{A}'$ is a translation plane derived from $\mathbf{A}$ (for the general theory of translation planes the reader may consult Dembowski [12] and Lüneburg [18]). $\mathbf{A}'$ may again be described by a pcp (as in the situation of 5.1), say, $\mathbf{U}'$, obtained from $\mathbf{U}_0$ by adding new components. It is well known that these components are in fact Baer subplanes of $\mathbf{A}$. The assertion that $\mathbf{U}_0$ is maximal is equivalent to saying that the dilatation group $H$ of $\mathbf{A}$ (which again also acts on $\mathbf{A}'$ by Lemma 3.1) does not fix all these Baer subplanes. But this is a situation which has recently been studied by Biliotti and Lunardon [5]. We remark that all Baer subplanes used in deriving a plane are always desarguesian (even if $\mathbf{A}$ is not a translation plane) by a result of Prohaska [25] (see also [18]). Let us now state our result:

6.1. THEOREM. *Let* $A'$ *be a plane derived from an affine translation plane of order* $q^2$ *and dimension* 2 *over its kernel* $K$. *Moreover, assume that the dilatation group* $H$ *of* $\mathbf{A}$ *does not fix all the Baer subplanes containing* *the origin of* $\mathbf{A}$ *used in constructing* $\mathbf{A}'$ *and that the Baer subplane defined by* $K$ *is one of the subplanes used. Then there exists a maximal partial* 1-*spread* $\mathbf{U}_0$ *of deficiency* $q-1$ *over* $GF(q)$ *such that* $\mathbf{D} = \mathbf{D}(\mathbf{U}_0)$ *is imbeddable into* $\mathbf{A}'$.

*Proof.* $\mathbf{A}$ may be obtained from a complete 1-spread $\mathbf{U}$ over $GF(q)$. Let $U_1, \ldots, U_{q^2+1}$ be the components of $\mathbf{U}$ and assume that $\mathbf{A}'$ has been constructed from $\mathbf{A}$ by replacing the net defined by the components $U_{q^2-q+1}, \ldots, U_{q^2+1}$. Furthermore let the Baer subplanes of $\mathbf{A}$ through the origin used in the construction of $\mathbf{A}'$ be $B_1, \ldots, B_{q+1}$. Then in fact $\mathbf{A}' = \mathbf{D}(\mathbf{U}')$ where $\mathbf{U}' = \{U_1, \ldots, U_{q^2-q}, B_1, \ldots, B_{q+1}\}$. Let $Q$ be a quasifield coordinatizing $\mathbf{A}$ and assume (by abuse of language, but without loss of generality) that its kernel is $K$. By a result of Ostrom [21] (cf. also Dembowski [12, p. 225]), $B_1, \ldots, B_{q+1}$ are precisely the $q+1$ point sets

$$C_a = \{(x, y) \colon x, y \in aK\} \tag{6.1a}$$

for $a \in Q^*$ (and $U_1, \ldots, U_{q^2-q}$ are the sets $\{(x, y) \colon y = xa\}$ for $a \in Q \backslash K$). Obviously $H$ fixes $C_1$; by a result of Biliotti and Lunardon [5], $H$ has to fix exactly 0, 2 or $q+1$ of the $B_i$. As $H$ fixes neither exactly 0 nor exactly $q+1$ of the $B_i$, we conclude that it fixes exactly two of them, say, $B_1$ and $B_2$. Then all other $B_i$'s are subgroups of the additive group of $V(4, q)$, but not subspaces.

Then $\mathbf{U}_0 = \{U_1, \ldots, U_{q^2-q}, B_1, B_2\}$ is the desired maximal partial 1-spread of $GF(q)$: Every further component $W$ would in particular be a transversal of the net $\mathbf{D}_1$ determined by $U_1, \ldots, U_{q^2-1}$. But a result of Ostrom [22] implies that the only transversals of $\mathbf{D}_1$ are the lines in the parallel classes determined by $U_{q^2-q+1}, \ldots, U_{q^2+1}$ and $B_1, \ldots, B_{q+1}$. We have already seen that $W \neq B_3, \ldots, B_{q+1}$; and also $W \neq U_i$ for $i = q^2 - q + 1, \ldots, q^2 + 1$ as each such $U_i$ intersects $B_1$ (and $B_2$) in $q$ points. ∎

6.2. *Remarks.* (i) That $H$ cannot fix exactly one of the $B_i$ is also a consequence of our Theorem 5.1.

(ii) There are many examples of derived translation planes $\mathbf{A}'$ which do not yield maximal partial 1-spreads: Clearly all $B_i$ will be fixed under $H$ if the derivation is actually effected by a regulus reversal (see, e.g., Dembowski [12, p. 225]). In particular, no sub-regular plane $\mathbf{A}'$ can occur in this way.

We will now show that the assumptions of Theorem 6.1 may be satisfied whenever $q$ is a prime power which is not a prime. This will also provide a simple alternative existence proof for such $q$ in the situation of (1.7b) to those of [7, 11 and 14].

6.3. THEOREM. *Let $q$ be any prime power which is not a prime. Then there exists a maximal partial* 1*-spread* $\mathbf{U}$ *of deficiency* $q-1$ *over* $GF(q)$ *for which* $\mathbf{D}(\mathbf{U})$ *is imbeddable.*

*Proof.* We shall use the semi-fields of Knuth exhibited in Dembowski [12, p. 241 (17)]. Thus let $\sigma \in \operatorname{Aut} GF(q)$ and let $f$, $g$ be non-zero elements of $GF(q)$ such that the polynomial $x^{\sigma+1} + gx - f$ is irreducible over $GF(q)$. Then a semi-field multiplication on $V(2, q)$ is given by (with 1, e a basis of $V(2, q)$)

$$(x + ye)(u + ve) = xu + yv^\sigma f + (xv + yu^\sigma + yv^\sigma g)\, e. \tag{6.3a}$$

Using the notation of the proof of 6.1 we will now check when a dilatation $\bar{\lambda}$ fixes a Baer subplane $C_a$ (given by (6.1a). Now using (6.3a) we here have

$$C_{x+ye} = \{(xu + yu^\sigma e, xw + yw^\sigma e) : u, w \in GF(q)\} \tag{6.3b}$$

and therefore

$$(C_{x+ye})^{\bar{\lambda}} = \{(\lambda xu + \lambda yu^\sigma e, \lambda xw + \lambda yw^\sigma e) : u, w \in GF(q)\} = C_{\lambda x + \lambda ye}. \tag{6.3c}$$

Hence $C_{x+ye}$ is fixed under $\bar{\lambda}$ iff there exists $\delta \in GF(q)^*$ with $\lambda x + \lambda ye = (x + ye)\delta = x\delta + y\delta^\sigma e$, i.e., iff $x = 0$ or $y = 0$ or $\lambda^\sigma = \lambda$. As $\sigma \neq 1$ we see that not all $C_a$ can be fixed by $H$ (in fact, only $C_1$ and $C_e$ are fixed by $H$, as it should be according to 6.1). Hence Theorem 6.1 may be applied to the plane defined by this semi-field. ∎

## 7. SOME NEW MAXIMAL PARTIAL $t$-SPREADS

As far as the author is aware no examples of maximal partial $t$-spreads (with $t = 2a + 1 \neq 1$) of deficiency $q^{a+1} - 1$ over $GF(q)$ are known at present. We will now exhibit examples of this type.

7.1. THEOREM. *Let $p$ be a prime and $r$ and $a$ be positive integers with $r \geqslant 2$. Then there exists a maximal partial* $(2a+1)$*-spread with deficiency* $p^{r(a+1)} - 1$ *over* $GF(p^r)$.

*Proof.* This result may be obtained as a consequence of the proof of Theorem 6.3. Defining $q = p^{r(a+1)}$ and taking $\sigma: x \to x^p$ we see that a component $C_a$ unequal $C_1$ or $C_e$ is fixed only by the dilatations induced from elements $\lambda$ of $GF(p)^*$. Hence none of these components are a subspace of $V(2a+2, p^r)$ and the same sort of argument as that given at the end of the proof of Theorem 6.1 shows that $\{U_1, \ldots, U_{q^2-q}, C_1, C_e\}$ is the desired maximal partial $(2a+1)$-spread. ∎

## 8. Concluding Remarks

We begin by observing that the new examples of maximal (translation) nets of small deficiency obtained here also yield corresponding new examples of maximal $(s, r; \mu)$-nets (which are structures where non-parallel "blocks" intersect in $\mu$ instead of 1 points). This follows from Theorem 4.2 of Jungnickel and Sane [17]; the reader is referred to this paper for a precise definition of an $(s, r; \mu)$-net too.

To conclude this paper, let us mention some open problems:

(i) The most important open problem on partial $t$-spreads is (in my opinion) the question of whether or not there are maximal partial 1-spreads of deficiency strictly smaller than $q-1$ (and similarly for partial $(2a+1)$-spreads). It is widely conjectured that such partial 1-spreads do not exist (e.g., by Bruen and Thas [11]). Theorem 6.1 certainly shows that examples of this type cannot be constructed by deriving a translation plane of order $q^2$ as described there. This may (together with 6.3) give some further support to the conjecture mentioned.

(ii) It would be interesting to know whether all partial 1-spreads of deficiency $q-1$ over $GF(q)$ are imbeddable (at least if they are constructed by Bruen's method sketched at the beginning of Section 6).

(iii) Are there any maximal partial 1-spreads of deficiency $q-1$ over $GF(q)$ wich are imbeddable into a non-derivable affine translation plane?

(iv) What is the precise deficiency of the maximal nets obtained from the unimbeddable nets of 3.5 and 5.2?

(v) Are there any maximal nets of deficiency strictly smaller than $s-1$ and order $s^2$? Are there maximal nets of small deficiency and order $s$ where $s$ is not a square or not even a prime power?

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